Complex Analysis: Resit Exam

Aletta Jacobshal 01, Friday 13 April 2018, 18:30–21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of each answer sheet and on the envelope.
- Use the ruled paper for writing the answers and use the blank paper as scratch paper. After finishing put your answers in the envelope. **Do NOT seal the envelope!** You must return the scratch paper and the printed exam (separately from the envelope).
- Solutions should be complete and clearly present your reasoning. **When you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.**
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (18 points)

(a) (3 points) Solve $z^3 + i = 0$ (you may express the solutions in Cartesian, trigonometric, or exponential form). Draw the solutions on the complex plane.

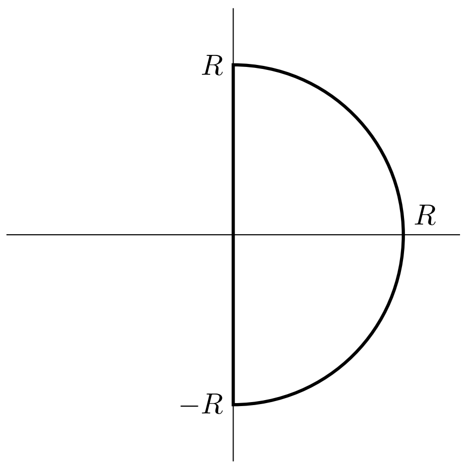
(b) (15 points) Evaluate

$$\mathrm{pv} \int_{-\infty}^{\infty} \frac{x}{x^3 + i}\, \mathrm{d}x$$

using the calculus of residues. *NB: This subquestion uses the result from subquestion (a) and will be graded* ***only if*** *subquestion (a) has been correctly answered and the result from subquestion (a) is properly used.*

Question 2 (15 points)

Apply Rouché's theorem with the type of contour C_R shown below to prove that the equation $z = 2 - e^{-z}$ has exactly one solution in the right half plane (that is, for $\mathrm{Re}(z) \geq 0$). Why must this solution be real?



Question 3 (15 points)

Represent the function

$$f(z) = \frac{z^2}{z-2},$$

(a) (8 points) as a Taylor series around 0 and give its radius of convergence;

(b) (7 points) as a Laurent series in the domain $|z| > 2$.

Question 4 (15 points)

Consider the function

$$f(z) = x + ax^2 - y^2 + iy + ibxy,$$

where $a, b \in \mathbb{R}$.

(a) (9 points) Find the values of a and b for which $f(z)$ is entire.

(b) (6 points) Let's call a_0 and b_0 respectively the (correct) values of a and b from subquestion (a). If $b = b_0$ but $a \neq a_0$ determine the subset of $\mathbb{C}$ where $f(z)$ is differentiable and the subset where it is analytic.

Question 5 (12 points)

Consider the function

$$f(z) = \frac{\cos(z) + 1}{(z-\pi)^2}.$$

(a) (9 points) Compute the Laurent series of $f(z)$ for $|z - \pi| > 0$.

(b) (3 points) Determine the singularities of $f(z)$ and their type (removable, pole, essential; if pole, specify the order).

Question 6 (15 points)

(a) (6 points) Find the most general harmonic polynomial $ax^2 + 2bxy + cy^2$, that is, determine the relations between the (real) constants a, b, c, so that the given polynomial is a harmonic function.

(b) (9 points) Determine the harmonic conjugate of the polynomial $ax^2 + 2bxy + cy^2$ for the a, b, c determined in subquestion (a). *NB: Finding a harmonic conjugate will only work if the correct relations between a, b, c have been determined. If finding the harmonic conjugate does not work you should check the correctness of your result in subquestion (a).*